Inhibition of chaos in Hamiltonian systems by periodic pulses

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It is shown that, depending on their amplitude, period, shape, and initial phase, a time-dependent periodic string of external driving pulses can suppress classical deterministic stochasticity. The analysis is based on a coupled pendulum–harmonic-oscillator system. Similar results are obtained by studying the behavior of the Lyapunov exponent from a simple recursion relation which models an unstable limit cycle affected by a periodic string of pulses.

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The dynamics of chaotic Hamiltonian systems, in both their classical and quantum versions, has attracted great interest in recent years [1,2]. Much work has been devoted to the study of kicked oscillators, where, in the overwhelming majority of cases, the periodic δ function has been used to model the periodic kicks [3]. Numerical (and analytical) results are easily obtained in these cases, and, in the quantum context, the results are believed to be qualitatively generic [4]. However, for classical dissipative systems it has recently been found that the dynamics of a generic periodically pulsed nonlinear oscillator is very sensitive to changes in the geometrical shape of the periodic pulses [5]: the system displays different types of bifurcations (homoclinic, Hopf) under suitable conditions when only the pulse shape is varied.

Another general setting, the possibility of controlling chaotic systems, has also inspired much recent theoretical and experimental work [6–14], due both to its intrinsic interest and to the many possible technological applications. Thus, Ott, Grebogi, and Yorke showed in Ref. [7] that one can change the motion of a chaotic dynamical system into periodic motion by controlling the system to stay near one of the many unstable periodic orbits embedded in the chaotic attractor by means of only weak time-dependent perturbations to an accessible system parameter. Also, Braiman and Goldhirst demonstrated in Ref. [12] the possibility of eliminating chaos by applying weak harmonic forcing via the example of a driven damped pendulum. However, little is known yet about the possible role of the shape of the weak, time-dependent perturbation causing this suppression of the chaotic dynamics.

Here it is proposed to study the inhibitory effect of a periodic sequence of pulses of equal amplitude on the chaotic behavior of a Hamiltonian system. The analysis will follow two lines.

First, I consider the well-known simple coupled oscillator model describing a plane pendulum coupled to a harmonic oscillator [15]. This Hamiltonian system admits chaotic behavior, and by virtue of having homoclinic orbits is amenable to the Melnikov method of analysis [15–17]. As has been well established, if the Melnikov function has simple zeros, then the stable and the unstable manifolds intersect transversely for weak perturbations, yielding homoclinic points. The resulting motion is so irregular that one can describe it as chaotic. I will show that, if an additional periodic string of pulses is applied, then, depending on their shape, period, initial phase, and amplitude, the Melnikov function can be prevented from admitting simple zeros, which implies that such periodic pulses inhibit the deterministic stochasticity.

Second, I analyze the behavior of the Lyapunov exponent (LE) from a simple recursion relation that models an unstable limit cycle affected by a periodic pulsatile perturbation. The results of this analysis are in qualitative agreement with those of the pulsed coupled pendulum–harmonic oscillator, so that the main conclusions seem to be generic.

Let us consider the following Hamiltonian for the coupled oscillator model [15,16]:

$$H(q,p,x,v)=F(q,p)+G(x,v)+\varepsilon H^{(1)}(q,p,x,v) , \quad (1)$$

where

$$F(q,p)=\tfrac{1}{2}p^2+(1-\cos q) , \quad (2)$$

$$G(x,v)=\tfrac{1}{2}(v^2+\omega^2x^2) , \quad (3)$$

and

$$H^{(1)}(q,p,x,v)=\frac{K}{2}(x-q)^2 , \quad (4)$$

denote the plane pendulum, the harmonic oscillator, and the coupling perturbation, respectively, and K, ε, and ω, are the coupling and perturbation parameters, and the angular frequency of the harmonic oscillator, respectively. The Hamiltonian perturbation $H^{(1)}$ destroys the integrability by introducing Smale horseshoes into the dynamics and hence the possibility of chaos. Following Guckenheimer and Holmes [cf. Eq. (4.8.48) of Ref. [15]], the Melnikov function for the perturbed Hamiltonian H, Eq. (1) is

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$$M(t_0)=\frac{2\pi\sqrt{2(h-2)}}{\omega}\operatorname{sech}\left[\pi\frac{\omega}{2}\right]\sin(\omega t_0)\ . \tag{5}$$

In calculating Eq. (5), the total energy of the homoclinic orbit had been taken as 2 and $H(q,p,x,v)=h$. Since $M(t_0)$ has simple zeros and is independent of ε, it may be concluded that, for $0<\varepsilon\ll 1$, one can have transverse intersection (and Smale horseshoes in the Poincaré map) when $h>2$.

Let us now look at the effect of a time-dependent periodic string of external solitonlike pulses represented by the perturbation $H^{(2)}$, of the same order $[O(\varepsilon)]$ as $H^{(1)}$:

$$H^{(2)}=Aq(1-\sqrt{1-m}\ )^{-1}[\mathrm{dn}(2Kt/T+\varphi;m)-\sqrt{1-m}\ ]\ . \tag{6}$$

Here dn is the Jacobian elliptic function (JEF) of parameter m, K is the elliptic integral of the first kind, and A,T, and φ $(0\leq\varphi\leq 2k)$ are the amplitude, and initial phase, respectively, of the pulses. The dn function modeling the periodic pulses is not an arbitrary choice but, quite to the contrary, is physically meaningful since the JEF's form the periodic solutions of the most-studied, nonlinear, integrable oscillators such as the Duffing, the pendulum, or the Helmholtz [2] oscillators.

Besides invariance under spatial reflection, $x\to -x,\ q\to -q,\ p\to p,\ v\to v,\ t\to t$, the undriven system ($H^{(1)}\neq 0$, $H^{(2)}\equiv 0$) possesses the time reversal symmetry $x\to x,\ q\to q,\ p\to -p,\ v\to -v,\ t\to -t$. For a general periodic driving, this twofold symmetry is destroyed and substituted by the discrete time-translation invariance under $t\to t+T$. In the particular case of a harmonic driving $[H^{(2)}=Aq\cos(\omega t)]$, the symmetry $f[t+(T/2)]=-f(t)$, restores a similar situation as in the undriven case: The system is now invariant under the generalized parity transformation P, $x\to -x,\ q\to -q,\ p\to p,\ v\to v,\ t\to t+(T/2)$ [18]. Observe that this can no longer be stated for the driving in (6) due to the nature of the dn function. [This is relevant on the quantum-mechanical level, where the generalized parity enables us to classify the Floquet functions into states of even and odd parity, respectively.]

Now the energy function

$$H=F(q,p)+G(x,v)+\varepsilon[H^{(1)}+H^{(2)}] \tag{7}$$

is no longer conserved and one has to consider [17] an equation for the time development of H in addition to the Hamiltonian equations of motion

$$\dot q=\frac{\partial F}{\partial p}+\varepsilon\frac{\partial H^{(1)}}{\partial p};\quad \dot p=-\frac{\partial F}{\partial q}-\varepsilon\frac{\partial H^{(1)}}{\partial q}-\varepsilon f\ ,$$
$$\dot\theta=\omega+\varepsilon\frac{\partial H^{(1)}}{\partial I};\quad \dot I=-\varepsilon\frac{\partial H^{(1)}}{\partial \theta}\ , \tag{8}$$

where

$$f=A(1-\sqrt{1-m}\ )^{-1}[dn(2Kt/T+\varphi;m)-\sqrt{1-m}\ ]\ , \tag{9}$$

and (I,θ) are the usual action angle variables defined by the canonical change of coordinates $x=(2I/\omega)^{1/2}\sin\theta, v=\omega(2I/\omega)^{1/2}\cos\theta$. Observe that, for fixed A, T, and φ, one can vary the shape of the pulses by changing the elliptic parameter m between 0 and 1. The special form of f is introduced for eliminating the steady push inherent in the dn function. Figure 1 shows three plots of the driving pulses (9) for different m values. If $m=0$, then $f=A\cos^2(\pi t/T)$, i.e., we recover the corresponding case of *harmonic* forcing. With increasing m, the width of the force becomes lower and lower (see Fig. 1), and for $m\simeq 1$ one has periodic sharply kicking forcing very close to the periodic δ function, but with finite width and amplitude as in real observed pulses (see, e.g., Ref. [19]). In the other limit we have $dn[2Kt/T;m\to 1]=0$, i.e., the pulse area tends to 0 if $m\to 1$, for $A,T=$const. Following Holmes and Marsden [16,20], by applying to (7) the classical reduction scheme [17] along with an average $\bar h$ instead of h, the Melnikov function obtained is

$$M_f(t_0)=(1/\omega^2)\left\{M(t_0)-\int_{-\infty}^{\infty}\left[\frac{\partial F}{\partial p}f\right]_{t-t_0}dt\right\}\ , \tag{10}$$

where $M(t_0)$ is given by (5) with the substitution for h by some average $\bar h$ appropriate for the time-dependent H [Eq. (7)]. (Note that, although there are several averaging procedures [17], there is no need to give $\bar h$ explicitly for our purposes.) Now, making use of $\partial F/\partial p=p$, the Fourier expansion for dn [21], and the homoclinic orbit associated with the F system

$$q(t)=2\arctan(\sinh(t))\ ,$$
$$p(t)=2\operatorname{sech}(t)\ , \tag{11}$$

for the integral in (10)

$$\int_{-\infty}^{\infty}\left[\frac{\partial F}{\partial p}f\right]_{t-t_0}dt=AP(t_0,\varphi;T,m)\ , \tag{12}$$

with

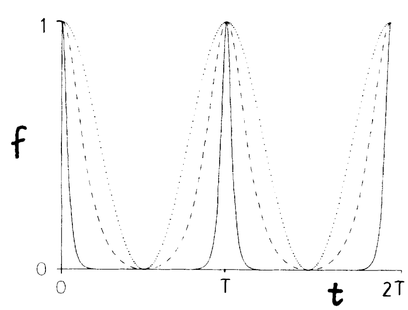

FIG. 1. Pulse function f, [Eq. (9)], for $A=$const, $T=$const, $\varphi=0$, and $m=0.5$ (dotted line), $m=0.999$ (dashed line), and $m=1-10^{-15}$ (solid line).

$$P(t_0,\varphi;T,m)=\frac{1}{1-\sqrt{1-m}}\times\left[\frac{\pi^2}{K}\sum_{n=-\infty}^{n=\infty}\operatorname{sech}\left[\frac{n\pi K'}{K}\right]\operatorname{sech}\left[\frac{n\pi^2}{T}\right]\times\cos\left[\frac{2n\pi t_0}{T}+\frac{n\pi\varphi}{K}\right]-2\pi\sqrt{1-m}\right]. \tag{13}$$

It is straightforward to demonstrate that $P(t_0,\varphi;T,m)$, regarded as a function of t_0 only, has maxima at $t_0=[n-(\varphi/2K)]T$, $n=0,1,\ldots$, and minima at

$$t_0=\left[n+\tfrac{1}{2}-\frac{\varphi}{2K}\right]T,\quad n=0,1,\ldots .$$

Hence,

$$P_{\max}(T,m)\geq P(t_0,\varphi;T,m)\geq P_{\min}(T,m)\geq 0. \tag{14}$$

with

$$P_{\max}(T,m)=\frac{1}{1-\sqrt{1-m}}\left[\frac{\pi^2}{K}\sum_{n=-\infty}^{n=\infty}\operatorname{sech}\left[\frac{n\pi K'}{K}\right]\times\operatorname{sech}\left[\frac{n\pi^2}{T}\right]-2\pi\sqrt{1-m}\right], \tag{15a}$$

$$P_{\min}(T,m)=\frac{1}{1-\sqrt{1-m}}\left[\frac{\pi^2}{K}\sum_{n=-\infty}^{n=\infty}\operatorname{sech}\left[\frac{n\pi K'}{K}\right]\times\operatorname{sech}\left[\frac{n\pi^2}{T}\right](-1)^n-2\pi\sqrt{1-m}\right], \tag{15b}$$

$$P_{\max}(T,m=0)=\pi\left[1+\operatorname{sech}\left[\frac{\pi^2}{T}\right]\right], \tag{15c}$$

$$P_{\min}(T,m=0)=\pi\left[1-\operatorname{sech}\left[\frac{\pi^2}{T}\right]\right], \tag{15d}$$

$$P_{\max}(T,m=1)=P_{\min}(T,m=1)=0 . \tag{15e}$$

A typical plot of $P_{\max}(m)$, $P_{\min}(m)$, with T const, is shown in Fig. 2. Therefore, one can recast Eq. (10) in the form

$$M_f(t_0)=(1/\omega^2)\{R(\bar{h},\omega)\sin(\omega t_0)-AP(t_0,\varphi;T,m)\} \tag{16}$$

where

$$R(\bar{h},\omega)=\frac{2\pi\sqrt{2(\bar{h}-2)}}{\omega}\operatorname{sech}\left[\frac{\pi\omega}{2}\right]. \tag{17}$$

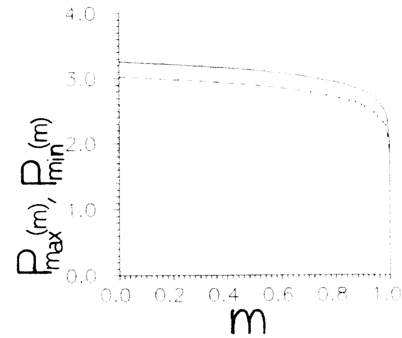

FIG. 2. Threshold functions $P_{\max}(m)$ (solid line) and $P_{\min}(m)$ (dashed line) versus m [Eqs. (15a) and (15b), respectively] in a generic situation, the remaining parameters being held constant.

Now, with $T_H\equiv 2\pi/\omega$ being the harmonic oscillator period, the following lemma gives the conditions under which the suppression of classical deterministic stochasticity is possible.

Lemma. Let $[4m+2(1-\varphi/K)]T=(4n+1)T_H$ for some positive integers m and n. Then $M_f(t_0)$ always has the same sign, specifically, $M_f(t_0)<0$, if and only if

$$R(\bar{h},\omega)<AP_{\min}(T,m) . \tag{18}$$

Proof. One has from Eq. (14)

$$\omega^2M_f(t_0)=[R(\bar{h},\omega)\sin(\omega t_0)-AP(t_0,\varphi;T,m)]\leq[R(\bar{h},\omega)\sin(\omega t_0)-AP_{\min}(T,m)] ,$$

and therefore condition (18) implies that $M_f(t_0)<0$. The converse follows from the existence of a value of t_0 such that

$$\sin(\omega t_0)=1;\quad P(t_0,\varphi;T,m)=P_{\min}(T,m) .$$

This is a consequence of the "selective" resonance condition of the lemma.

Remarks. First, observe that a requirement is $\varphi=IK$, I integer ($I<2m+1$), for the above resonant condition to be fulfilled for some positive integers m and n. It is worth mentioning that while such a resonant constraint is not required for suppressing Hamiltonian chaos, it needs to be imposed for Eq. (18) also to represent a necessary condition for the inhibition of Hamiltonian stochasticity. Second, for fixed $\bar{h}$ and ω, if the pulses are very narrow ($m\simeq 1$) it is clear from Eq. (15e) and Fig. 2 that condition (18) is not readily fulfilled since $\varepsilon A\ll 1$. In other words, with the same period and amplitude, wide pulses can suppress chaos more easily than sharp ones.

It must be stressed that adding a pulsatile external field has distinct effects on different orbits. However, as the relevance of the coexistence of infinitely many periodic unstable orbits is today well accepted, and the situation is considered equivalent to steady chaos, one may compare the above results with those from the model of an unstable limit cycle affected by a small periodic pulse perturbation:

$$X_{n+1}=(\mu+\varepsilon f_n)X_n \ , \tag{19}$$

with $\mu > 1$, $f_n=(1-\sqrt{1-m})^{-1}[dn(2Kn/T;m)-\sqrt{1-m}\,]$. A similar recursion relation with f_n a harmonic function was studied in Ref. [12]. Note that $\langle f_n\rangle=(1-\sqrt{1-m})^{-1}[\pi/2K-\sqrt{1-m}\,]$ angular brackets denoting an average over n. When $\varepsilon=0$, the fixed point x is unstable. To study the effect of the weak pulsatile perturbation, let us calculate the LE for $\varepsilon\neq 0$:

$$\lambda=\mathrm{Re}\langle \ln(\mu+\varepsilon f_n)\rangle \ . \tag{20}$$

If the limit cycle is weakly unstable then $\mu=1+|\delta|$, $|\delta|\ll 1$. In this situation, for small ε, Eq. (20) becomes $\lambda=|\delta|-\varepsilon P'(m)+O(\delta^2,\varepsilon^2)$, with

$$P'(m)=\frac{1}{1-\sqrt{1-m}}\left[\frac{\pi}{2K}-\sqrt{1-m}\right] . \tag{21}$$

A plot of $P'(m)$ is shown in Fig. 3 (note the great similarity with the curves of Fig. 2). When $|\delta|<\varepsilon P'(m)$, the LE λ is negative, i.e., x is stable. On the contrary, if $|\delta|>\varepsilon P'(m)$, λ is positive and x is unstable. In order to clarify the pure effect of pulse shape on the reduction of instabilities (positive LE), consider that we have an initial state characterized by $\varepsilon=\varepsilon^*, m=m^*\sim 1$ such as $|\delta|>\varepsilon^*P'(m^*)$. Then, by decreasing m, one decreases the LE λ which, in some cases, may become negative, thus stabilizing x.

In summary, I have shown by way of a coupled pendulum–harmonic-oscillator example that, depending on their amplitude, period, initial phase, and geometrical shape, a time-dependent periodic string of external driving pulses can reduce and suppress deterministic stochasticity. That the period and initial phase of the pulses satisfy a condition of *selective* resonance with the period of the harmonic oscillator has allowed us to form a sufficient and necessary condition for the dynamics to become regular. The condition was only sufficient without this selective resonance. It was also demonstrated that the inhibitory effect of the pulsatile external field is very sensitive to changes in the geometrical form of the pulses: wide pulses suppress stochasticity more readily than narrow ones. Finally, similar results were found using a simple model recursion relation, leading one to conjecture their generic nature.

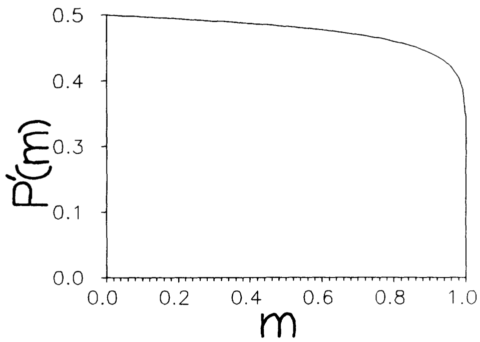

FIG. 3. Function $P'(m)$ versus m [Eq. (21)].

This study, which is intended solely as an analytical survey, should encourage someone to investigate the problem in a numerical form. A related topic which deserves further investigation is the stability of subharmonic orbits.

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